**R N S INSTITUTE OF TECHNOLOGY**

**DEPARTMENTOF MATHEMATICS**

**Channasandra, Bangalore- 560 098**

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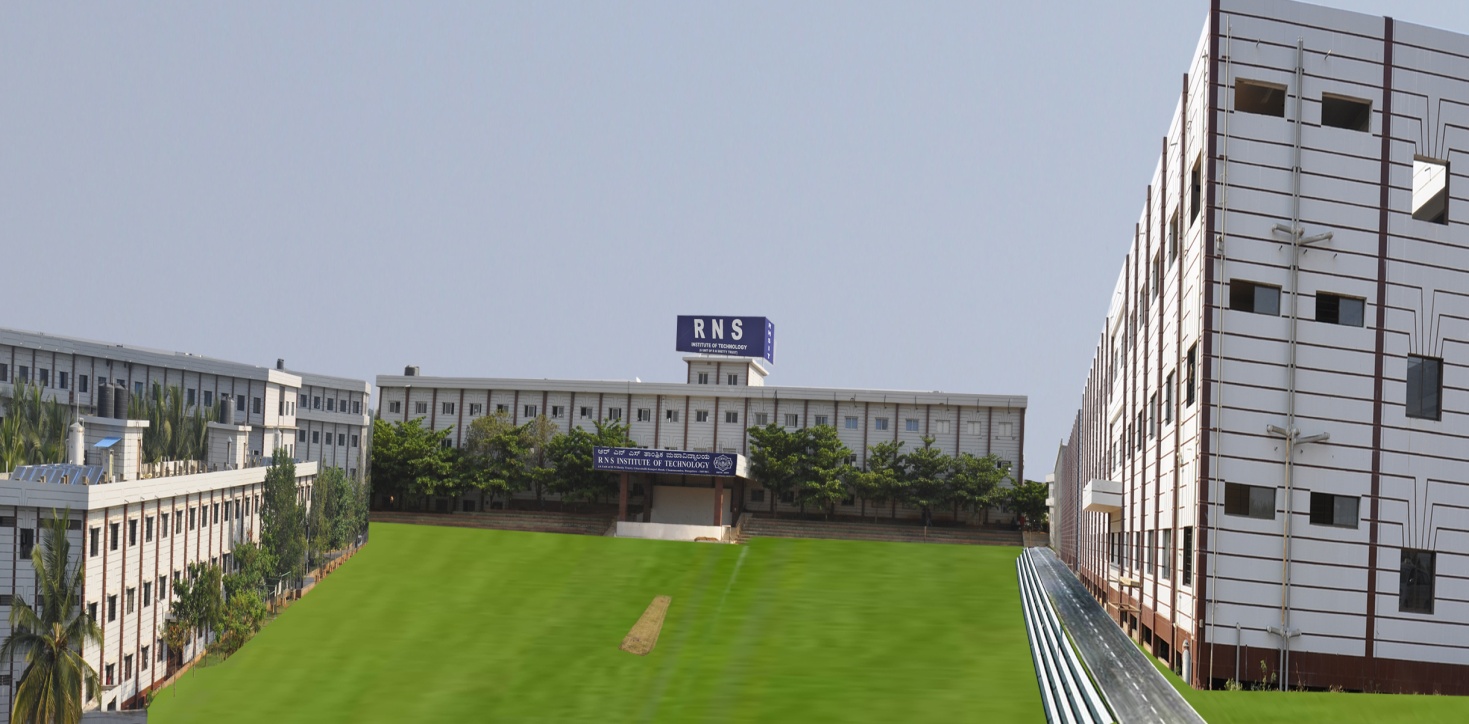
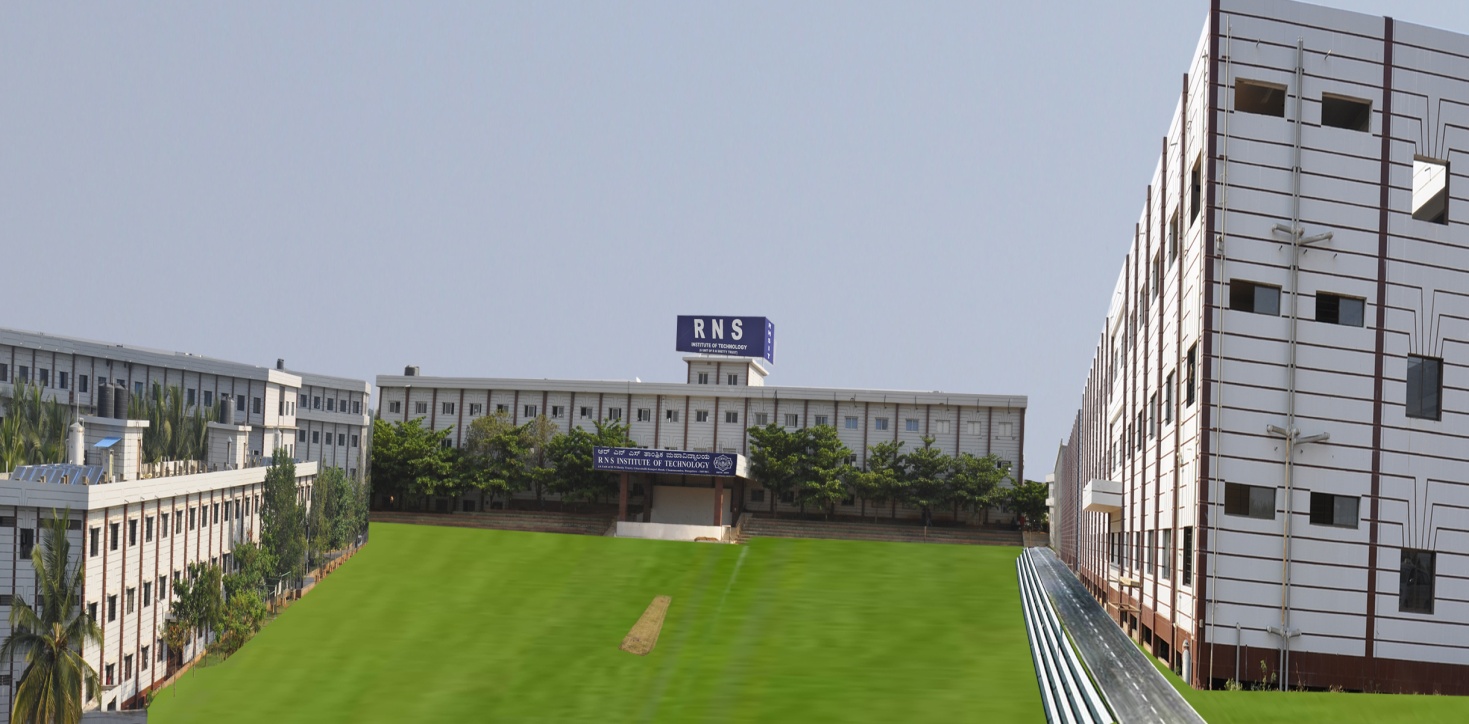
**STUDY MATERIALS**

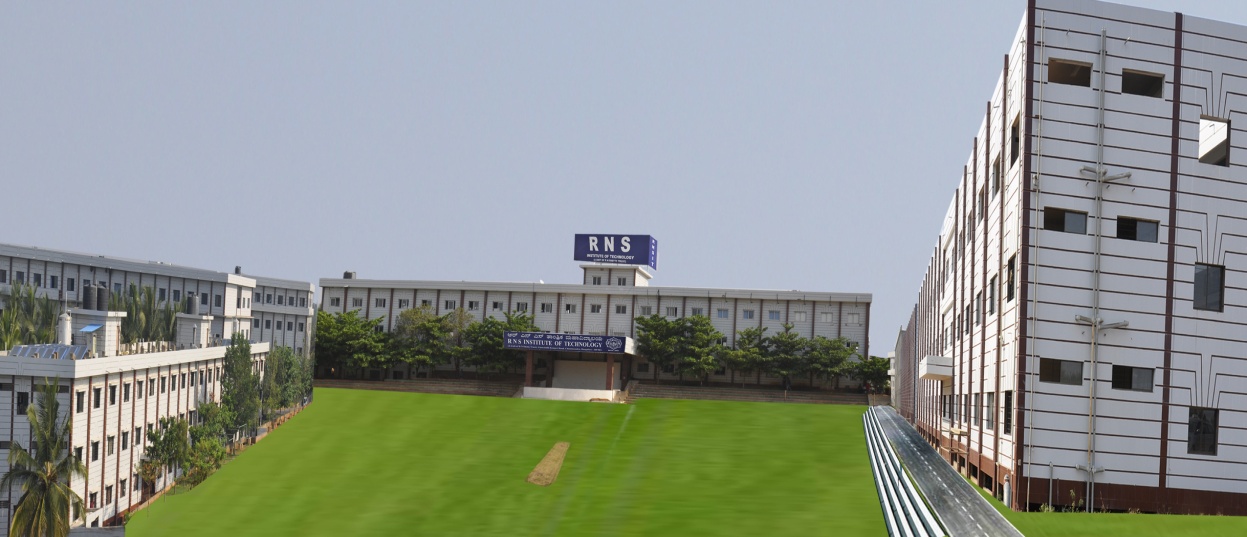
**FOR**

**MATHEMATICS**

**VTU NEW SYLLABUS**

**MODULE-V**

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**LINEAR ALGEBRA**

Linear algebra comprises of the theory and application of linear system of equations,linear transformations and eigen value problems.In linear algebra a systematic use of matrices and to a lesser extent determinants and their properties.

* Determinants were fist introduced for solving linear systems and have important engineering applications in systems of differential equations,electrical networks,eigen value problems and so on.It can simplify many complicated electrical and mechanical systems by expressing them in terms of determinants.
* Matrices can be used in solving homogenous and non homogenous equations. Used to find area of triangle. In vectors for cross product. Hermitian or complex matrix in multidimensional analysis.
* Before computer graphics, the science of optics used matrix mathematics to account for reflection and for refraction.
* Matrix arithmetic helps us calculate the electrical properties of a circuit, with voltage, amperage, resistance, etc.
* In mathematics, one application of matrix notation supports graph theory. In an adjacency matrix, the integer values of each element indicates how many connections a particular node has.
* The field of probability and statistics may use matrix representations. A probability vector lists the probabilities of different outcomes of one trial. A stochastic matrix is a square matrix whose rows are probability vectors. Computers run Markov simulations based on stochastic matrices in order to model events ranging from gambling through weather forecasting to quantum mechanics.
* Stochastic matrices and Eigen vector solvers are used in the page rank algorithms which are used in the ranking of web pages in Google search.
* Matrices and their inverse matrices are used for a programmer for coding or encrypting a message.

**Cayley discovered matrices in the year 1860s but it was not until the twentieth centuary that engineers heard of them.**



Arthur Cayley

Elementary transformations (operations)

The following are the elementary transformations or elementary operations of a matrix:

(i) Interchange of two rows or two columns.

(ii) Multiplication of each element of a row or column by a non – zero number *k.*

(iii) Addition of *k* times the elements of a row (or column) to the corresponding elements of another row (or column), 

NOTE: An elementary transformation (operation) applied to rows only is called a **row transformation (operation)** and an elementary transformation (operation)

We use the following symbols for elementary transformations (operations):

|  |  |
| --- | --- |
| Elementary transformation (operation) | Symbol |
| Interchanging the ith and jth rows |  |
| Interchanging the ith and jth columns |  |
| Changing the ith row to *k* times the ithe row |  |
| Changing the ithcolumn *k* times the ith column |  |
| Changing the ith row to the sum of *k* times the ith row and *p* times the jth row |  |
| Changing the ith column to the sum of *k* times the ith column and *p* times the jth column |  |

Equivalent Matrix: If a matrix B is obtained from a matrix A by one or more elementary transformations (operations), then the matrix B is said to be equivalent to A. Two equivalent matrices A and B can be written as 

Reduction of Matrix: In matrix algebra, by using elementary transformations, we can reduce the matrix to the (1) Echelon form and (2) Normal form

(1) Echelon form:

A non-zero matrix A is said to be in row echelon form if the following conditions hold:

(i) All the zero rows are below the non-zero rows.

(ii) The first non-zero element in each non-zero row after the first row appears in a column that lies to right of the first non-zero element in any preceding row.

For example, the following matrices are in row echelon form

Note: The echlon form of a matrix is obtained by the application of successive row operations to the given matrix

2. Normal form:

A non-zero matrix A is said to be in normal form if it can be reduced to any one of the following forms by elementary transformations (operations).



Where is an identity matrix of order 

For example, the following matrices are in normal forms

Note: The normal form of a matrix is obtained by the application of successive row or column or both operations to the given matrix

RANK OF A MATRIX

Let A be any  matrix. It has square sub-matrices of different orders. The determinants of these square sub-matrices are called minors of A. If all minors of order  are zero but there is at least one non-zero minor of order *r*, then *r* is called the rank of A.

Symbolically, rank of written as 

Some consequences:

From the definition of the rank of a matrix A, we have the following consequences:

(i) If A is a null matrix, then

[ every minor of A has zero value]

(ii) If A is not a null matrix, then 

(iii) If A is a non-singular  matrix, then 

[ is the largest minor of A]

If  is the  unit matrix, then 

(iv) If A is an  matrix, then  minimum of *m* and *n*.

(v) If all minors of order *r* are equal to zero, then 

Note: (1) The rank of a matrix is not altered by elementary transformations applies to the matrix.

(2) The rank of a matrix is not altered when the matrix is multiplies by a non-singular matrix.

(3) A matrix and its transpose have the same rank.

Determination of Rank of the Matrix:

To determine the rank of a matrix A, we adopt the following different method by using elementary transformations:

Method – 1: Enumeration

Start with the highest order minor/minors of A. let their order be *r*. If any one od them is non-zero, then .

If all of them are zero, start with minors of next lower order  and so on till you get a non-zero minor. The order of that minor is the rank of A.

This method usually involves a lot of computational work since we have to evaluate several determinants.

Note: This method is impracticable for matrices of higher order

Method -2: Echelon form

The number of non-zero rows present in an Echelon form of a matrix of matrix A is called its rank.

Method -3: Normal form

If a matrix A of order  can be reduced to any one of the following forms:

, , ,  (normal forms)

Then the rank of the matrix A is *r*

Problems

1. Find the rank of the matrix  by elementary transformations.

Solution: Let  be a given matrix of order 

 (The smaller of 2 and 4)

The second order minor 



2. Find the rank of the matrix  by elementary transformations.

Solution: Let  be a given matrix of order 

Operating 



Since second and third rows are identical, the third order minor is zero.



Again operating 



The second order minor 



3. Find the rank of the matrix  by elementary transformations.

Solution: Let  be a given matrix of order 

 (The smaller of 3 and 4)

Operating 



Operating 



Operating 



Since the last row contains only zeros, the fourth order minor is zero.



The second order minor 

4. Find the rank of the matrix  by elementary transformations.

Solution: Let  be a given matrix of order  

Operating 



Since the last row contains only zeros, the fourth order minor is zero.



Operating 



Operating 



The third order minor 



5. Find the rank of the matrix  by elementary transformations.

Solution: Let  be a given matrix of order  

Operating 



Operating 



Operating 



Operating 





The fourth order minor is non –zero.



6. Find the rank of the matrix  by elementary transformations.

Solution: Let  be a given matrix of order , 

Operating 



Operating 



Operating 



Every third order minor is zero, hence 

The second order minor 



7. Find the rank of the matrix by elementary transformations.

Solution: Let  be a given matrix of order 

Operating 



Operating 



Operating 



Operating 



The matrix A is in row echelon form having four non zero rows



8. Find the rank of the matrix  by reducing it to the echelon form.

Solution:

Let 

 Operating 

 Again operating 

This is echelon form, it contains two non-zero rows.



9. Find the rank of the matrix  by reducing it to the echelon form.

Solution:

Let  Operating

 Operating 

 Again operating 



This is echelon form, it contains two non-zero rows.



10. Find the rank of the matrix  by reducing it to the echelon form.

Solution:

Let 

Operating 



Operating 



Operating 



Operating 



Operating 



Operating 



Operating 



Operating 



Again operating 



This is normal form of order 3



11.Find the rank of the matrix  by reducing it to the normal form

Solution: Let 

Operating 



Operating 



Operating 



Operating 



Operating 



Operating 



Operating 



This is normal form of order 4



12. Find the value of , so that the rank of the matrix  is three.

Solution:

Let 

Given that ,

 fourth order minor is zero



Operating 



 (Expanding along first column)





 or 

 or 

Thus, the rank of the given matrix is 3 whenever  or 

13. Find the value of *k*, so that the rank of the matrix  is two

Solution:

Let 

Given that 



Operating 

 (expanding along first column)







Thus, the rank of the given matrix is 2 whenever 

EXERCISE

Find the rank of the following matrices by elementary transformations

1.  2.  3. 4. 

5.  6.  7.  8. 

9.  10. 

Answers

(1) 1 (2) 2 (3) 2 (4) 2 (5) 2 (6) 4 (7) 3 (8) 3 (9) 3 (10) 2

SOLUTION OF SYSTEM OF LINEAR EQUATIONS

Simulataneous linear equations occur in various Engineering problems. The following numerical methods give the solution of system of linear equations which are well – suited for computing machines.

(1) Gauss Elimination method

(2) Gauss – Jordan method

(3) Gauss – Seidel method

GAUSS ELIMINATION METHOD

In this method, the unknowns are eliminated successively and the system is reduced to an upper trianglular system from which the unknown are found by back substitution.

Working Procedure :

Consider the system of equations







The above system of equations can be written in matrix from as





Where 

Consider the augmented matrix 

By using the element to make the elements  and  zero by elementary row transdormations. This reduces to the following form



Again by using the element  to make the element  and zero by elementary row transdormations. This reduces to the following form



This form is called upper triangular form

From this, we get the new system of equations as







We get the value of *z* from the last equation and by back substitution we get *y* and *x*. This method of finding the solution of the given system of linear equations is called Gauss elimination method

This method can be generalized to system of *n* equations in *n* unknowns.

Problems:

1. Apply Gauss elimination method to solve the system of equations







Solution: The augmented matrix associated with the given system of equation is

 Apply the operation 





The associated system of equation is 





Solving these equations by back substitution we get, 





And 



Therefore the solution for the given system is 

2. Apply Gauss elimination method to solve the system of equations







Solution: The augmented matrix associated with the given system of equation is

 Apply the operation 





The associated system of equation is 





Solving these equations by back substitution we get, 





And 



Therefore the solution for the given system is 

3. Apply Gauss elimination method to solve the system of equations







Solution: The given system of equation is 





Rewriting the given system we get ,  and 

The associated augmented matrix is

 Apply the operation 







The associated system of equation is 





Solving these equations by back substitution we get, 





And 



Therefore the solution for the given system is 

4. Apply Gauss elimination method to solve the system of equations







Solution: The given system of equation ,  and 

The associated augmented matrix is

 Apply the operation 





The associated system of equation is 





Solving these equations by back substitution we get, 





And 



Therefore the solution for the given system is 

5. Apply Gauss elimination method to solve the system of equations









Solution: The given system of equation , ,   
  and 

The associated augmented matrix is



Apply the operation 









The associated system of equation is 







Solving these equations by back substitution we get, 









And 

Therefore the solution for the given system is 

6. Apply Gauss elimination method to solve the system of equations







Solution: The given system of equation is ,  and 

The augmented matrix associated with the given system of equation is

 Apply the operation 









The associated system of equation is 





Solving these equations by back substitution we get, 





And 



Therefore the solution for the given system is 

3. Gauss-Seidel method: The Gauss-elimination and Gauss-Jordan method of solving non-homogenous systems of linear equations. These methods gives the exact solutions for the systems and are called Direct methods. In many situations, it is tedious to obtain exact solutions and we therefore resort to find only approximate solutions. One of the ways of finding an approximate solution is to known as the Gauss-Seidel method.

Consider the system of equations

 (1)

The given system equation (1) is diagonally dominant i.,e, the diagonal coefficients  are not zero and are numerically large compared to sum of the other two coefficients in the respective equations

The system of equation (1) may be rewritten as

 (2)

Let  is an initial approximate solution of the system (1)

Substitute  for y and z in the first equation in (2), we get a value of x which we denoted by  and is taken as first approximate value of x.

We substitute  for x and z in the second equation in (2), we get a value of y which is denoted by  and is taken as first approximate value of y.

We substitute  for x and y in the third equation in (2), we get a value of z denoted by .

The values of x, y, z obtained is the first iterates of the solution. Proceeding in the same way we obtain successive iterates.

This process is stopped when the desired order of approximation is reached or when two successive iterates are identical or nearly identical.

The final values so obtained constitute an approximate solution of the system (1). This method can be generalized to a diagonally dominant system of n(>3) equations in n unknowns.

Problems:

1. Employing the Gauss – seidel method, solve the system of equations:

,  and 

The given equations may be rewritten as

 (i)

 (ii)

 (iii)

We take the rough (initial) approximations of  as  .

First approximation

In the first approximation, equations (i) – (iii) yield\*







Second approximation

In the second approximation, equations (i) –(iii) yield\*







Third approximation, equations (i) – (iii) yield\*







We observe that the third approximate solution is identical with second approximate solution up to the second decimal. We therefore take



As a solution of the given system, correct to two decimal places. (We check that this is indeed the exact solution).

2. Using the Gauss – Seidel method, solve the equations

,  and  (VTU 2013)

Solution: The given equations may be rewritten as

 (1)

 (2)

 (3)

Let the initial approximation of x, y, z as 

First approximation

In the first approximation, y = 0 and z = 0 in (i)



Put x = 0.85, and z = 0 in (2),



Put x = 0.85 and y = -1.0275 in (3),



Second approximation

In the second approximation, put y = -1.0275 and z=1.01098 in (1)

Put x = 1.0025, and z = 1.01098 in (2),



Put x = 1.0025 and y = -0.9998 in (3),



Third approximation

In the Third approximation, put y = -0.9998 and z=0.9998 in (1)

Put x = 0.99997, and z = 0.9998 in (2),



Put x = 0.99997 and y = -1.0000 in (3),



We observe that after the third approximate solution 

3. Using the Gauss – Seidel method, solve the equations

,  and  by taking (x, y, z) = (1, 1, 1) as the initial approximate solution.

Solution: The given equations may be rewritten as

 (1)

 (2)

 (3)

Let the initial approximation of x, y, z as 

First approximation

In the first approximation, y = 1 and z = 1 in (i)



Put x = 0.8889, and z = 1 in (2),



Put x = 0.8889 and y = 1.6111 in (3),



Second approximation

In the second approximation, put y = 1.6111 and z=1.1966 in (1) 

Put x = 0.9131, and z = 1.1966 in (2),



Put x = 0.9131 and y = 1.6480 in (3),



Third approximation

In the Third approximation, put y = 1.6480 and z = 1.1946 in (1) 

Put x = 0.9176 and z = 1.1946 in (2),



Put x = 0.9176 and y = 1.6472 in (3),



We observe that after the third approximate solution

Fourth approximation

In the Fourth approximation, put y = 1.6472 and z = 1.1954 in (1) 

Put x = 0.9174 and z = 1.1954 in (2),



Put x = 0.9174 and y = 1.6473 in (3),



We observe that after the third approximate solution



4. Using the Gauss – Seidel method, solve the equations ,   
  and . Carryout 4 iterations, by taking (x, y, z) = (1, 0, 3) as the initial approximate solution.

Solution: The given equations may be rewritten as

 (1)

 (2)

 (3)

Let the initial approximation of x, y, z as 

First approximation

In the first approximation, y = 0 and z = 3 in (i)



Put x = 1.8, and z = 3 in (2),



Put x = 1.8 and y = 1.8 in (3),



Second approximation

In the second approximation, put y = 1.8 and z = 2.92 in (1) 

Put x = 1.096, and z = 2.92 in (2),



Put x = 1.096 and y = 2.016 in (3),



Third approximation

In the Third approximation, put y = 2.016 and z = 2.9744 in (1) 

Put x = 0.9987 and z = 2.9744 in (2),



Put x = 0.9987 and y = 2.0131 in (3),



Fourth approximation

In the Fourth approximation, put y = 2.0131 and z = 2.9950 in (1) 

Put x = 0.9958 and z = 2.9950 in (2),



Put x = 0.9958 and y = 2.0036 in (3),



We observe that after the fourth approximate solution



5. Using the Gauss – Seidel method, solve the equations

,  and 

Solution: The given equations may be rewritten as

 (1)

 (2)

 (3)

Let the initial approximation of x, y, z as 

First approximation

In the first approximation, y = 0 and z = 0 in (i)



Put x = 1.1445, and z = 0 in (2),



Put x = 1.1445 and y = 1.8459 in (3),



Second approximation

In the second approximation, put y = 1.8459 and z = 1.8206 in (1) 

Put x = 0.9876, and z = 0 in (2),



Put x = 0.9876 and y = 1.4119 in (3),



Third approximation

In the Third approximation, put y = 1.4119 and z = 1.9566 in (1) 

Put x = 1.0517 and z = 1.9566 in (2),



Put x = 1.0517 and y = 1.3692 in (3),



We observe that after the third approximate solution 

6. Using the Gauss – Seidel method, solve the equations

,  and  carry out four iterations, taking the initial approximation to the solution as (2, 3, 2).

Solution: The given equations may be rewritten as

 (1)

 (2)

 (3)

Let the initial approximation of x, y, z as 

First approximation

In the first approximation, y = 3 and z = 2 in (i)



Put x = 2.5556, and z = 2 in (2),



Put x = 2.5556 and y = 3.5111 in (3),



Second approximation

In the second approximation, put y = 3.5111 and z = 1.9247 in (1) 

Put x = 2.5133, and z = 1.9247 in (2),



Put x = 2.5133 and y = 3.5381 in (3),



Third approximation

In the Third approximation, put y = 3.5381 and z = 1.950 in (1) 

Put x = 2.4332, and z = 1.950 in (2),



Put x = 2.4332 and y = 3.5701 in (3),



Fourth approximation

Put y = 3.5701 and z = 1.9259 in (1) 

Put x = 2.4261, and z = 1.9259 in (2),



Put x = 2.4261 and y = 3.5728 in (3),



We observe that after the fourth approximate solution 

EIGEN VALUES AND EIGEN VECTORS:

Let *A* be a square matrix of *n*,  be a scalar and *I* be the unit matrix of order *n* then the equation  is called characteristic equation or Eigen equation of A.

The roots of this equation are called Characteristic roots or Eigen values of the matrix A. corresponding to each value  (real or complex) and a non-zero column matrix X of order n such that , then  is called Eigen value of A and X is called an Eigen vector of the matrix A corresponding to an Eigen value . That is  gives the system of homogeneous linear equations having a non-zero solution  which is called an Eigen vector.

Note: The characteristic equation of a third order square matrix A can be obtained without expanding  by using the rule 

Where sum of the diagonal elements of A and sum of the minors of the diagonal elements of A and determinant of A.

Problems:

1. Find the Eigen values and Eigen vectors of the matrix 

Solution: The characteristic equation is 







 are the Eigen values of the matrix A.

Let  be the Eigen vector corresponding to the value , then 



 (1)

Case (i): put  in (1),





If we choose , then 

Case (ii): put  in (1),







Case (iii): put  in (1),







Thus for the Eigen value  the corresponding Eigen vectors are 

2. Find the Eigen values and Eigen vectors of the matrix  (VTU 2014)

Solution: The characteristic equation is 







 are the Eigen values of the matrix A.

Let  be the Eigen vector corresponding to the value , then 



 (1)

Case (i): put  in (1),

 (2)

Using the method of cross multiplication for first two equations in (2)





 is the Eigen vector corresponding to the Eigen value 

Case (ii): put  in (1),

 (3)

Using the method of cross multiplication for first two equations in (3)





 is the Eigen vector corresponding to Eigen value 

Case (iii): put  in (1),

 (4)

Using the method of cross multiplication for first two equations in (4)





 is the Eigen vector corresponding to Eigen value 

Thus for the Eigen value  the corresponding Eigen vectors are 

3. Find the Eigen values and Eigen vectors of the matrix 

Solution: The characteristic equation is





The characteristic equation is  (1)

Where sum of the diagonal elements of A = 1 + 5 + 1 = 7

sum of the minors of the diagonal elements of A



and 

therefore the equation (1) gives 

By solving this equation  are the Eigen values of the matrix A.

Let be the Eigen vector corresponding to the value , then





 (1)

Case (i): put  in (1),

 (2)

Using the method of cross multiplication for first two equations in (2)





 is the Eigen vector corresponding to the Eigen value 

Case (ii): put in (1),



 (3)

Using the method of cross multiplication for first two equations in (3)





 is the Eigen vector corresponding to Eigen value



Case (iii): put  in (1),

 (4)

Using the method of cross multiplication for first two equations in (4)





 is the Eigen vector corresponding to Eigen value 

Thus for the Eigen value  the corresponding Eigen vectors are 

4. Find the Eigen values and Eigen vectors of the matrix 

Solution: The characteristic equation is





The characteristic equation is  (1)

Where sum of the diagonal elements of A = -2 + 1 + 0 = -1



sum of the minors of the diagonal elements of A





and 

therefore the equation (1) gives 

By solving this equation  are the Eigen values of the matrix A.

Let be the Eigen vector corresponding to the value , then





 (1)

Case (i): put  in (1),

 (2)

Using the method of cross multiplication for first two equations in (2)





 is the Eigen vector corresponding to the Eigen value 

Case (ii): put  in (1),

 (3)

All the equations in (3) are same i.,e 

We choose two of  arbitrarily.

For , taking 

 is the Eigen vector corresponding to Eigen value 

Case (iii): put  in (1), this is same of case (ii)

taking 

 is the Eigen vector corresponding to Eigen value 

Thus for the Eigen value  the corresponding Eigen vectors are 

Rayleigh’s power method:

This method is used to determine the largest Eigen value (dominant Eigen value) and the corresponding Eigen vector of a square matrix.

Working rule:

Let A be a given square matrix

* First we choose an Eigen vector *X0* in the form 
* Compute the matrix product  which is a column matrix.
* Take the largest element as the common factor (normalisation) to obtain of the form .
* Then compute the matrix product  and again put it in the form  by normalization.
* This iterative process is continued till two consecutive iterative values of  and X are same upto the desired degree of accuracy.
* The values so obtained are respectively the largest Eigen value and the corresponding Eigen vector of the given square matrix A.

Problems:

1. Find the largest Eigen value and the corresponding Eigen vector of the matrix   
  by Rayleigh’s power method. (VTU 2014)

Solution: Let  be the initial Eigen vector













Hence the largest Eigen value is  and the corresponding Eigen vector is .

2. Find the largest Eigen value and the corresponding Eigen vector of the matrix   
  by Rayleigh’s power method. (VTU 2011)

Solution: Let  be the initial Eigen vector















Hence the largest Eigen value is  and the corresponding Eigen vector is .

3. Find the largest Eigen value and the corresponding Eigen vector of the matrix   
  by Rayleigh’s power method. Perform five iterations. (VTU 2013)

Solution: Let be the initial Eigen vector













Hence the largest Eigen value is  and the corresponding Eigen vector is .

4. Find the largest Eigen value and the corresponding Eigen vector of the matrix   
  by Rayleigh’s power method.

Solution: Let be the initial Eigen vector















Hence the largest Eigen value is  and the corresponding Eigen vector is .

5. Find the largest Eigen value and the corresponding Eigen vector of the matrix   
  by Rayleigh’s power method. (VTU 2008)

Solution: Let be the initial Eigen vector









Hence the largest Eigen value is  and the corresponding Eigen vector is .

6. Find the largest Eigen value and the corresponding Eigen vector of the matrix   
  by Rayleigh’s power method. Perform 5 iteration by taking the initial Eigen vector as 

Solution: Let  be the initial Eigen vector













Hence the largest Eigen value is  and the corresponding Eigen vector is .

7. Find the largest Eigen value and the corresponding Eigen vector of the matrix   
  by Rayleigh’s power method. Perform 5 iteration, by taking initial Eigen vector as [1, 1, 1]T. (VTU 2013)

Solution: Let  be the initial Eigen vector













Hence the largest Eigen value is  and the corresponding Eigen vector is .

8. Find the largest Eigen value and the corresponding Eigen vector of the matrix   
  by Rayleigh’s power method. Perform 8 iteration

Solution: Let  be the initial Eigen vector

















Hence the largest Eigen value is  and the corresponding Eigen vector is .

9. Find the largest Eigen value and the corresponding Eigen vector of the matrix   
  by Rayleigh’s power method. Perform five iterations, by taking

[0, 0, 1]T as the initial approximation to the Eigen vector.

Solution: Let  be the initial Eigen vector











Hence the largest Eigen value is  and the corresponding Eigen vector is .

Exercise

Find the numerically largest (dominant) Eigen value and the corresponding Eigen vector of the following matrices by the Power method.

1.  2.  3.  4. 

5.  6. 

Answers:

1.  2. 

3.  4. 

5.  6. 

**Linear transformation:**

Let a point *P(x, y)* in a plane transforms to the point  under reflection in the co-ordinate axes or reflection in the line *y = x* or rotation of OP through an angle  about the origin, etc.

Then the co-ordinates of  can be expressed in terms of those of *P* by the linear relations of the form  and  this can be represented in matrix form as 

Such transformations are called linear transformations in two dimensions. i.e., *Y = AX* gives a linear transformation  in two dimensions.

Similarly, the linear transformations in three dimensions is of the form , and  which in matrix form is 

i.e., *Y = AX* gives a linear transformation  in three dimensions.

In general, the relation *Y = AX*, where 

defines a linear transformation which carries any vector *X* into another vector *Y* over the matrix *A* called the linear operator of the transformation. This transformation is called linear because , for all values of a and b.

Remark:

* If the transformation matrix *A* is non-singular i.e.,  then the linear transformation is called non-singular or regular.
* If the transformation matrix *A* is singular i.e.,  then the linear transformation is called singular.
* For a non-singular transformation *Y = AX*, since *A* is non-singular, *A-1*exists and we can write the inverse transformation, which carries the vector *Y* back into the vector *X* as *X = A -1Y.*

Note: If a transformation from  is given by *Y = AX* and another transformation from  is given by *Z = BY*, then the transformation from  is given by *Z = BY = B(AX) = (BA)X*.

Definition: OTHOGONALTRANSFORMATION

The linear transformation *Y = AX* is said to be orthogonal transformation if the matrix *A* is an orthogonal matrix.

**Note:**

* A square matrix *A* is said to be orthogonal if .
* If *A* is an orthogonal matrix, then .
* The inverse of an orthogonal matrix is orthogonal and its transpose is also orthogonal i.e., if *A* is orthogonal then *A-1* and  are also orthogonal.

Problems:

1. Find the matrix of the linear transformation that transforms the pair  to the   
 pair .

Solution: The given transformation transforms  to  where   
 

Therefore the matrix of the linear transformation is 

2. Show that the linear transformation  is regular linear transformation. Find the inverse of this transformation.

Solution: Given 

Therefore the matrix of the given transformation is 



 A is non singular.

Thus the given transformation is regular.

By using  or by elementary transformations we find that   
 

Therefore the inverse of the given transformation is *X = A-1Y*



3. Show that the linear transformation , and  is   
 a regular linear transformation. Find the inverse of this transformation. (VTU 2013)

Solution: The matrix form of the linear transformation is 

Therefore the matrix of the given transformation is 



 A is non singular.

Thus the given transformation is regular.

By using  or by elementary transformations we find that   
 

Therefore the inverse of the given transformation is *X = A-1Y*



i.e., , and 

4. Show that the linear transformation , and  is a regular linear transformation. Find the inverse of this transformation.

Solution: The matrix form of the linear transformation is 

Therefore the matrix of the given transformation is 



 A is non singular.

Thus the given transformation is regular.

By using  or by elementary transformations we find that   
 

Therefore the inverse of the given transformation is *X = A-1Y*



i.e., , and 

5. Show that the linear transformation , and  is a regular linear transformation. Find the inverse of this transformation.

Solution: The matrix form of the linear transformation is 

Therefore the matrix of the given transformation is 



 A is non singular.

Thus the given transformation is regular.

By using  or by elementary transformations we find that   
 

Therefore the inverse of the given transformation is *X = A-1Y*



i.e., , and 

6. Show that the linear transformation ,  and is a regular linear transformation. Find the inverse of this transformation.

Solution: The matrix form of the linear transformation is 

Therefore the matrix of the given transformation is 



 A is non singular.

Thus the given transformation is non-singular.

By using  or by elementary transformations we find that   
 

Therefore the inverse of the given transformation is *X = A-1Y*



i.e., , and 

7. A transformation from the variables  to  is given by *Y = AX* and another transformation from  to  is given by *Z = BY* where  obtain the transformation from  to .

Solution: we have *Y = AX* and  *Z = BY*







Therefore the linear transformation from  to  is given by

,  and .

8. Represent each of the linear transformations  by the use of matrices and find the composite transformation which expresses  in terms of .

Solution: The matrix form of the linear transformations are 

And 

We have 



Therefore the required composite transformation is



Reduction of a matrix to Diagonal form: